

ON THE MULTIPLICATIVE ERDŐS DISCREPANCY PROBLEM

MICHAEL COONS

ABSTRACT. As early as the 1930s, Pál Erdős conjectured that: *for any multiplicative function $f : \mathbb{N} \rightarrow \{-1, 1\}$, the partial sums $\sum_{n \leq x} f(n)$ are unbounded.* In this paper, after providing a counterexample to this conjecture, we consider completely multiplicative functions $f : \mathbb{N} \rightarrow \{-1, 1\}$ as well as a class of similar multiplicative functions f satisfying

$$\sum_{p \leq x} f(p) = c \cdot \frac{x}{\log x} (1 + o(1)).$$

We prove that if $c > 0$ then the partial sums of f are unbounded, and if $c < 0$ then the partial sums of μf are unbounded. Extensions of this result are also given.

1. INTRODUCTION

Erdős [2] asked the following question, sometimes known as the Erdős Discrepancy Problem. “Let $f(n) = \pm 1$ be an arbitrary number theoretic function. Is it true that to every c there is a d and an m for which

$$(1) \quad \left| \sum_{k=1}^n f(kd) \right| > c ?$$

Inequality (1) is one of my oldest conjectures.” (This particular quote is taken from a restatement of the conjecture in [3, p.78]. See also [4] and [5].) Erdős offered 500 dollars for a proof of this conjecture. Erdős [2, p.293] wrote in 1957 that this conjecture is twenty-five years old, placing its origin at least as far back as the early 1930s. In [2, 3, 4], Erdős also stated a multiplicative form of his conjecture.

Conjecture 1.1 (Erdős). *Let $f(n) = \pm 1$ be a multiplicative function, (i.e., $f(ab) = f(a)f(b)$, when $\gcd(a, b) = 1$). Then*

$$(2) \quad \limsup_{x \rightarrow \infty} \left| \sum_{n \leq x} f(n) \right| = \infty;$$

that is, the partial sums of f are unbounded.

Erdős added in [3] that “clearly (2) would follow from (1) but as far as I know (2) has never been proved. Incidentally (2) was also conjectured by Tchudakoff.”

Conjecture 1.1 as stated is not true, and while this may be known to others in this field, there seems to be no account of it in the literature.

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For a counterexample, consider the multiplicative function g defined by $g(1) = 1$, and on prime powers by

$$(3) \quad g(p^k) = \begin{cases} -1 & \text{if } p = 2 \text{ and } k \geq 1 \\ 1 & \text{if } p \neq 2 \text{ and } k \geq 1. \end{cases}$$

Then g is periodic with period 2 and for all $n \geq 1$ we have $g(2n) = -1$ and $g(2n-1) = 1$. Thus

$$\sum_{n \leq x} g(n) = \begin{cases} 1 & \text{if } [x] \text{ is odd} \\ 0 & \text{if } [x] \text{ is even,} \end{cases}$$

and so

$$\limsup_{x \rightarrow \infty} \left| \sum_{n \leq x} g(n) \right| = 1.$$

It may very well be the case that the function g defined above is the only counterexample to Conjecture 1.1, but at least at this point, we can say that this is the only known counterexample.

Along with Conjecture 1.1, Erdős [2] conjectured a result on the mean values of multiplicative functions. A number-theoretic function $f : \mathbb{N} \rightarrow \mathbb{C}$ has a *mean value*, denoted $M(f)$, provided the limit

$$(4) \quad M(f) := \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n)$$

exists. Erdős [2, 3] (among others) conjectured that any multiplicative function taking the values ± 1 has a mean value; this is usually called the Erdős–Wintner Conjecture. In 1961, Delange [1] characterized those functions with positive mean value, and in 1967, Wirsing [10] gave a complete solution to this conjecture, as well as the extension to all complex-valued multiplicative functions f satisfying $|f| \leq 1$. This was later refined by Halász [9] in 1968. We state the result here only for those functions with which we are directly concerned.

Theorem 1.2 (Delange, Wirsing, Halász). *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a multiplicative function. If*

$$(5) \quad \sum_{p \leq x} \frac{1 - f(p)}{p}$$

is bounded then $M(f)$ exists and is positive, and if (5) is unbounded then $M(f) = 0$.

We note that the ideas of Theorem 1.2 have been generalized by many authors, including Granville and Soundararajan [7, 8] and Goldmakher [6]. In these works the authors use properties of a generalization of (5) to give some new results concerning sums of certain types of Dirichlet characters. The generalization of (5) is usually made by considering a special multiplicative function g (e.g., a Dirichlet character) and comparing it to the multiplicative function of interest f (e.g., a Dirichlet character) by means of investigating the asymptotics of

$$\sum_{p \leq x} \frac{1 - \Re(f\bar{g}(p))}{p}.$$

This sum can be thought of as a metric [7], and in some sense measures how g mimics f ; this terminology was introduced in [6].

In contrast to this “mimicry metric,” we consider the asymptotics of

$$\sum_{p \leq x} \frac{c - f(p)}{p}$$

for c not necessarily equal to 1. By considering sums like this, we are able to give the following result toward Conjecture 1.1.

Theorem 1.3. *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a multiplicative function such that there is some $k \geq 1$ with $f(2^k) = 1$. Suppose that for some $c \in [-1, 1]$ we have*

$$\sum_{p \leq x} f(p) = c \cdot \frac{x}{\log x} (1 + o(1)).$$

If $c > 0$ then the partial sums of f are unbounded, and if $c < 0$ the partial sums of μf are unbounded.

Some extensions of this theorem are given in Section 4, including some instances of the case $c = 0$. In Section 2, we show that this theorem is true for completely multiplicative functions without the assumption that there is some $k \geq 1$ with $f(2^k) = 1$.

2. COMPLETELY MULTIPLICATIVE FUNCTIONS

If a multiplicative function $f : \mathbb{N} \rightarrow \{-1, 1\}$ has positive mean value, then clearly the partial sums of f are unbounded; they are asymptotic to $M(f) \cdot x$. The triviality leaves when we consider functions with $M(f) = 0$.

Theorem 2.1. *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a completely multiplicative function (i.e., $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{N}$) and suppose that $c \in [-1, 1)$. If*

$$\sum_p \frac{c - f(p)}{p} < \infty,$$

then the mean value of f exists and is equal to 0.

Proof. This follows from Theorem 1.2 in a very straightforward way. We need only note that

$$\begin{aligned} \sum_{p \leq x} \frac{1 - f(p)}{p} &= \sum_{n \leq x} \frac{1 - c + c - f(p)}{p} \\ &= (1 - c) \sum_{n \leq x} \frac{1}{p} + \sum_{n \leq x} \frac{c - f(p)}{p} = (1 - c) \log \log x + O(1). \quad \square \end{aligned}$$

To prove Theorem 1.3, we will first prove the result for completely multiplicative functions $f : \mathbb{N} \rightarrow \{-1, 1\}$. The bulk of the work is taken up by the following lemma.

Lemma 2.2. *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a completely multiplicative function. Suppose that $c \in [-1, 1]$ is nonzero and*

$$\sum_p \frac{c - f(p)}{p} < \infty.$$

If $c > 0$ then the partial sums of f are unbounded, and if $c < 0$ the partial sums of μf are unbounded.

Proof. Suppose firstly that $c > 0$. To give the desired result, it is enough to show that

$$\lim_{x \rightarrow \infty} \sum_{n \leq x} \frac{f(n)}{n} = \infty.$$

To this end, note that for $\sigma > 1$ we have

$$\begin{aligned} (6) \quad \log F(\sigma) &= \log \sum_{n \geq 1} \frac{f(n)}{n^\sigma} = - \sum_p \log \left(1 - \frac{f(p)}{p} \right) \\ &= \sum_p \sum_{k \geq 1} \frac{f(p)^k}{kp^{k\sigma}} = \sum_p \frac{f(p)}{p^\sigma} + \sum_p \sum_{k \geq 2} \frac{f(p)^k}{kp^{k\sigma}} = \sum_p \frac{f(p)}{p^\sigma} + O(1), \end{aligned}$$

where the $O(1)$ term is valid for $\sigma > 1/2$. Since

$$\sum_p \frac{c - f(p)}{p} < \infty,$$

we have that

$$(7) \quad \sum_{p \leq x} \frac{f(p)}{p} = c \log \log x + O(1).$$

The condition that $c > 0$ ensures that

$$\lim_{s \rightarrow 1^+} \sum_p \frac{f(p)}{p^s} = \infty,$$

and so the divergence of $\log F(\sigma)$ at $\sigma = 1$ occurs because $\lim_{\sigma \rightarrow 1^+} F(\sigma) = \infty$.

In the light of (6) it must be the case that

$$(8) \quad \lim_{x \rightarrow \infty} \sum_{n \leq x} \frac{f(n)}{n} = \infty.$$

Thus we have that

$$\limsup_{x \rightarrow \infty} \left| \sum_{n \leq x} f(n) \right| = \infty.$$

For if not, there is a real number $M > 0$ such that $\left| \sum_{n \leq x} f(n) \right| < M$, and by partial summation, we would then have that

$$\sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{x} \sum_{n \leq x} f(n) + \int_1^x \left(\sum_{n \leq t} f(n) \right) \frac{dt}{t^2} = O \left(\int_1^x \frac{dt}{t^2} \right) = O(1),$$

which contradicts (8).

Now suppose that $c < 0$. In this case, instead of $F(\sigma)$, we consider the function $1/F(\sigma)$. Running through the above argument gives

$$(9) \quad -\log F(\sigma) = - \sum_p \frac{f(p)}{p^\sigma} + O(1),$$

where again the $O(1)$ term is valid for $\sigma > 1/2$. Similar to the above, using the assumption of the lemma, we have that

$$(10) \quad - \sum_{p \leq x} \frac{f(p)}{p} = |c| \log \log x + O(1),$$

which in turn gives, due to (9) that

$$\lim_{\sigma \rightarrow 1^+} \frac{1}{F(\sigma)} = \infty.$$

This implies that

$$\lim_{x \rightarrow \infty} \sum_{n \leq x} \frac{\mu(n)f(n)}{n} = \infty,$$

which using a similar argument as the case $c > 0$, give that

$$\limsup_{x \rightarrow \infty} \left| \sum_{n \leq x} \mu(n)f(n) \right| = \infty.$$

This completes the proof of the lemma. \square

Our proof of the main theorem follows from the similar result for completely multiplicative functions. Using partial summation we have the following theorem.

Theorem 2.3. *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a completely multiplicative function. Suppose that for some $c \in [-1, 1]$ we have*

$$\sum_{p \leq x} f(p) = c \cdot \frac{x}{\log x} (1 + o(1)).$$

If $c > 0$ then the partial sums of f are unbounded, and if $c < 0$ the partial sums of μf are unbounded.

Proof. This follows directly from Lemma 2.2. The condition

$$\sum_{p \leq x} f(p) = c \cdot \frac{x}{\log x} (1 + o(1))$$

gives by partial summation that

$$\begin{aligned} \sum_{p \leq x} \frac{f(p)}{p} &= \frac{1}{x} \sum_{p \leq x} f(p) + \int_1^x \left(\sum_{p \leq t} f(p) \right) \frac{dt}{t^2} \\ &= c \cdot \frac{1}{\log x} (1 + o(1)) + c \int_1^x \frac{1}{t \log t} (1 + o(1)) dt \\ (11) \quad &= c \log \log x (1 + o(1)). \end{aligned}$$

Note that the proof of the lemma follows from the divergent behavior of $\sum_{p \leq x} \frac{f(p)}{p}$ in both (7) and (10), and that this divergence is satisfied by (11). Thus using (11) in the place of (7) and (10) is enough to prove Lemma 2.2, and thus the condition (11) implies the result of the theorem. \square

3. EXTENSION TO MULTIPLICATIVE FUNCTIONS

The results of the previous section are extendable to multiplicative functions $f : \mathbb{N} \rightarrow \{-1, 1\}$ with the added condition that there is some $k \geq 1$ with $f(2^k) = 1$. In this section, by relating a multiplicative function $f : \mathbb{N} \rightarrow \{-1, 1\}$ to a related completely multiplicative function, we are able to deduce Theorem 1.3 as a corollary to Theorem 2.3. This is obtained via the following lemma.

Lemma 3.1. *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a multiplicative function such that there is some $k \geq 1$ with $f(2^k) = 1$. Then*

$$F(\sigma) = \sum_{n \geq 1} \frac{f(n)}{n^\sigma} = \Pi(\sigma) \cdot \prod_p \left(1 - \frac{f(p)}{p^\sigma}\right)^{-1} \quad (\sigma > 1),$$

where

$$\Pi(\sigma) = \prod_p \left(1 + \sum_{k \geq 2} \frac{f(p^k) - f(p^{k-1})f(p)}{p^{k\sigma}}\right).$$

Moreover, there is a $\sigma_0(f) \in (0, 1)$ such that $\Pi(\sigma)$ is absolutely convergent for $\sigma > \sigma_0(f)$.

Proof. Note that if f is multiplicative, then for $\sigma > 1$ we have using the Euler product for its generating Dirichlet series that

$$\begin{aligned} F(\sigma) &:= \sum_{n \geq 1} \frac{f(n)}{n^\sigma} = \prod_p \left(1 + \frac{f(p^2)}{p^{2\sigma}} + \frac{f(p^3)}{p^{3\sigma}} + \dots\right) \\ &= \prod_p \left(1 - \frac{f(p)}{p^\sigma}\right)^{-1} \cdot \prod_p \left(1 + \sum_{k \geq 2} \frac{f(p^k) - f(p^{k-1})f(p)}{p^{k\sigma}}\right). \end{aligned}$$

It remains to show that $\Pi(\sigma)$ is absolutely convergent for $\sigma > \frac{\log \varphi}{\log 2}$. Firstly, note that for each prime p we have

$$\begin{aligned} (12) \quad &1 + \sum_{k \geq 2} \frac{f(p^k) - f(p^{k-1})f(p)}{p^{k\sigma}} \\ &\geq \max \left\{ 1 + \sum_{k \geq 2} \frac{f(2^k) - f(2^{k-1})f(2)}{2^{k\sigma}}, 1 - \frac{2}{3^\sigma(3^\sigma - 1)} \right\}. \end{aligned}$$

To ensure that none of the factors of the product $\Pi(\sigma)$ is zero, we will show that the right-hand side of (12) is greater than zero for $\sigma > \frac{\log \varphi}{\log 2}$. Now if $f(2^k) = 1$ for all $k \geq 1$, then

$$1 + \sum_{k \geq 2} \frac{f(2^k) - f(2^{k-1})f(2)}{2^{k\sigma}} = 1 > 0,$$

regardless of the range of σ . Thus we may suppose that $f(2^k) \neq 1$ identically. Then, using our assumption, since $f(2^k) = 1$ for at least one $k \geq 1$, we have for some $k \geq 2$ that $f(2) \neq f(2^k)$. Rephrased, this means that there is a $k \geq 3$ such that $f(2^{k-1})f(2) = -1$. Denote

$$k_0 := \min\{k \geq 3 : f(2^{k-1})f(2) = -1\}.$$

Then

$$1 + \sum_{k \geq 2} \frac{f(2^k) - f(2^{k-1})f(2)}{2^{k\sigma}} \geq 1 - 2 \sum_{k \geq 2} \frac{1}{2^{k\sigma}} + \frac{2}{2^{k_0\sigma}} = 1 - \frac{2}{2^\sigma(2^\sigma - 1)} + \frac{2}{2^{k_0\sigma}}.$$

Note that for $\sigma > 0$ and $k_0 \geq 2$, the function

$$1 - \frac{2}{2^\sigma(2^\sigma - 1)} + \frac{2}{2^{k_0\sigma}}$$

is continuous and increasing. Also at $\sigma = 1$ we have

$$1 - \frac{2}{2^1(2^1 - 1)} + \frac{2}{2^{k_0}} = \frac{2}{2^{k_0}} > 0,$$

so that by continuity and the fact that $k_0 \geq 3$, there is some minimal $\alpha := \alpha(k_0) \in (0, 1)$ such that for $\sigma > \alpha$ we have

$$1 - \frac{2}{2^\sigma(2^\sigma - 1)} + \frac{2}{2^{k_0\sigma}} > 0.$$

Also, we have that

$$3^{2\sigma} - 3^\sigma - 2 > 0$$

for $\sigma > \frac{\log 2}{\log 3}$ by the quadratic formula. Since $3^{2\sigma} - 3^\sigma - 2 > 0$ precisely when $1 - \frac{2}{3^\sigma(3^\sigma - 1)} > 0$, combining this with the above, we have that each of the terms of the product $\Pi(\sigma)$ is positive for all

$$\sigma > \sigma_0(f) := \max \left\{ \alpha, \frac{\log 2}{\log 3} \right\}.$$

Since this maximum is strictly less than one, the only thing left to show is that the sum $\sum_p \sum_{k \geq 2} \frac{f(p^k) - f(p^{k-1})f(p)}{p^{k\sigma}}$ is absolutely convergent. We have that

$$\sum_p \left| \sum_{k \geq 2} \frac{f(p^k) - f(p^{k-1})f(p)}{p^{k\sigma}} \right| \leq \sum_p \sum_{k \geq 2} \frac{2}{p^{k\sigma}} = 2 \sum_p \frac{1}{p^\sigma} \cdot \frac{1}{p^\sigma - 1},$$

which is convergent when $\sigma > 1/2$, proving the lemma. \square

It is worth remarking that assuming that $f(2^k) = 1$ for some $k \geq 1$ ensures that we are not considering the counterexample g defined in (3).

We now give the proof of Theorem 1.3 as a corollary to Theorem 2.3.

Proof of Theorem 1.3. Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a multiplicative function such that $f(2^k) = 1$ for some $k \geq 1$, and denote $F(\sigma) = \sum_{n \geq 1} \frac{f(n)}{n^\sigma}$. By Lemma 3.1, we have that

$$F(\sigma) = \Pi(\sigma)F_c(\sigma),$$

where $\Pi(\sigma)$ is defined by as in Lemma 3.1 and $F_c(\sigma)$ is the generating Dirichlet series for the completely multiplicative function $f_c : \mathbb{N} \rightarrow \{-1, 1\}$ defined by $f_c(p) = f(p)$ for all primes p . Similar to (6), we have that

$$\log F(\sigma) = \log \Pi(\sigma) + \log F_c(\sigma) = \sum_p \frac{f(p)}{p^\sigma} + O(1),$$

since $\Pi(\sigma) > 0$ for $\sigma \geq 1$. If $c > 0$, then considering the proof of Theorem 2.3 for $F_c(\sigma)$ gives that

$$\lim_{\sigma \rightarrow 1^+} F_c(\sigma) = \infty,$$

and so

$$\lim_{\sigma \rightarrow 1^+} F(\sigma) = \infty,$$

which in turn gives that the partial sums $\sum_{n \leq x} f(n)$ are unbounded.

If $c < 0$ we just consider the proof of Theorem 2.3 for the function $1/F(\sigma)$, and use the equation

$$\log \frac{1}{F(\sigma)} = -\log F(\sigma) = -\log \Pi(\sigma) - \log F_c(\sigma)$$

to yield the result. \square

4. WEAKENING OF HYPOTHESES AND FURTHER EXTENSIONS

In Theorem 2.3 we can replace the condition

$$(13) \quad \sum_p \frac{c - f(p)}{p} < \infty$$

with something considerably weaker.

Note that assumption (13) is given so that we may use an asymptotic of the form

$$\sum_{p \leq x} \frac{f(p)}{p} = c \log \log x + O(1),$$

for nonzero $c \in [-1, 1]$. In the case of positive c we can weaken the condition to

$$(14) \quad \lim_{x \rightarrow \infty} \sum_{p \leq x} \frac{f(p)}{p} = \infty,$$

and in the case of negative c we can weaken the condition to

$$(15) \quad \lim_{x \rightarrow \infty} \sum_{p \leq x} \frac{f(p)}{p} = -\infty.$$

Then if (14) holds we have that $\sum_{n \leq x} f(n)$ is unbounded, and if (15) holds we have that $\sum_{n \leq x} \mu(n)f(n)$ is unbounded. As far as “density conditions” the above limits are satisfied when we take

$$\sum_{p \leq x} f(p) = \frac{c \cdot x}{\log x \log_2 x \cdots \log_k x} (1 + o(1)),$$

where $\log_j x$ denotes $\log \log \cdots \log x$ with “log” written j times, k is any nonnegative integer, and $c \neq 0$ is taken to be positive or negative depending on the desired case; this is easily seen via partial summation.

We can do a little in the case that $c = 0$. Indeed, all we really need is to have for some $\sigma > 1/2$ that either

$$(16) \quad \lim_{x \rightarrow \infty} \sum_{p \leq x} \frac{f(p)}{p^\sigma} = \infty \quad \text{or} \quad \lim_{x \rightarrow \infty} \sum_{p \leq x} \frac{f(p)}{p^\sigma} = -\infty.$$

In this case, the same method gives the following theorem, though consideration of the limits in (16) directly gives a more exact result.

Theorem 4.1. *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a completely multiplicative function, $\sigma > 1/2$, k a nonnegative integer, and suppose that*

$$\sum_{p \leq x} f(p) = \frac{c \cdot x^\sigma}{\log x \log_2 x \cdots \log_k x} (1 + o(1)).$$

If $c > 0$ then the partial sums of f are unbounded, and if $c < 0$ then the partial sums of μf are unbounded.

As discussed above, the proof of Theorem 4.1 follows exactly the same as that of Theorem 2.3, and as such we omit it for fear of sounding redundant. Theorem 1.3 can be generalized similarly, but with the added assumptions that both $f \neq g$ for g as defined in (3), and that $\sigma > \sigma_0(f)$ as defined in the proof of Lemma 3.1.

5. CONCLUDING REMARKS

Functions satisfying (13) for positive c are, in some sense, large. In fact, since $F(\sigma)$ are divergent at $\sigma = 1$, we have, using an obvious abuse of notation, at least that

$$\sum_{n \leq x} f(n) \gg x^{1-\varepsilon}$$

for any $\varepsilon > 0$. Probably this can be improved, but our original purpose was to just prove the unboundedness of partial sums. Indeed, using the terminology of Goldmakher [6], we should have that the function $f(n)$ mimics the function $c^{\Omega(n)}$ and the partial sums of this function are quite large; we have

$$\sum_{n \leq x} c^{\Omega(n)} \geq \sum_{p \leq x} c = c \cdot \pi(x).$$

As an extension of the results for negative c , it would be nice if one could show that since $\sum_{n \leq x} \mu(n)f(n)$ is unbounded, so is $\sum_{n \leq x} f(n)$. We suspect that one may have to consider cases whether or not the Riemann hypothesis holds. Nonetheless, since we have

$$\sum_{n \leq x} \mu(n)f(n) \gg x^{1-\varepsilon}$$

for any $\varepsilon > 0$ and the partial sums of μ are not too small, $\sum_{n \leq x} \mu(n) \neq O(x^{1/2})$, something may be able to be done in this case. Indeed, we conjecture that in this case one should have at least that

$$\sum_{n \leq x} f(n) \gg x^{1/2-\varepsilon}$$

for any $\varepsilon > 0$. Towards something like this we have tried to factor $F(s)$ in an enlightening way (to find a singularity at $s = 1/2$, but to no avail. We note that one has for any such series $F(s)$ and $c \in [-1, 0)$, that

$$F(s) = \left(\frac{\zeta(2s)}{\zeta(s)} \right)^{|c|} \frac{e^{\frac{P(2s)}{2}} D(s)}{\zeta(2s)^{\frac{|c|}{2}}} \cdot \exp \left[- \sum_p \frac{c - f(p)}{p^s} \right],$$

where $P(s)$ is the prime zeta function and the function $D(s)$ is absolutely convergent for $\Re(s) > 1/3$.

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UNIVERSITY OF WATERLOO, DEPT. OF PURE MATH., WATERLOO, ON, N2L 3G1, CANADA
E-mail address: `mcoons@math.uwaterloo.ca`